國立臺灣大學 年度基礎學科免修認證考試試題

題號：2011
科目：微積分1

※ 注意：請於試卷上「非選擇題作答區」標明題號並依序作答。

※ 禁止使用計算機

考試須知：

- ▶ 不能使用計算機，電子辭典及個人自備之計算紙。
- ▶ 無論計算或證明題，皆應詳述過程、理由；如未寫出詳細過程，一律不給分。
- ▶ 將答案寫於試卷，並標示正確的題號。

1. Evaluate the following limits or show that they do not exist.

   (a) (8 pts) $\lim_{h\to 0} \dfrac{f(1+h)+f(1-h)-2f(1)}{2h^2}$ where $f(x) = \ln\left(\tan^{-1}\left(\dfrac{1+x}{2}\right)\right)$.

   (b) (8 pts) $\lim_{x\to 0} \dfrac{1-\sqrt{1+x}}{\sqrt{1-\cos 2x}}$.

   (c) (8 pts) $\lim_{x\to\infty} (f(3x)-f(x))\sin(\dfrac{1}{x})$ where $f(x)$ is a differentiable function and $\lim_{x\to\infty} f'(x) = 2$.

2. Let $f(x) = \begin{cases} (\cos 2x)^{\frac{1}{x}} & \text{, for } x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\} \\ a & \text{, for } x = 0 \end{cases}$, where a is a constant.

   (a) (8 pts) Find the constant a such that $f(x)$ is continuous at $x = 0$.

   (b) (12 pts) Suppose that $f(x)$ is continuous at $x = 0$. Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.

3. Consider the function $f(x) = \tan^{-1}(e^x) + e^x$.

   (a) (8 pts) Show that f is a one-to-one function.

   (b) (10 pts) Let $g(x) = f^{-1}(x)$, the inverse function of $f(x)$. Write down the linear approximation of $g(x)$ at $x = 1 + \frac{\pi}{4}$.

   (c) (2 pts) Use the linear approximation from part (b) to estimate the value of $g\left(1 + \dfrac{\pi}{5}\right)$.

4. Consider the function $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$.

   (a) (10 pts) Find $f'(x)$ and $f''(x)$.

   (b) (10 pts) Find all the asymptotes of $y = f(x)$.
   (`Hint` : You may use the identity $A + B = \dfrac{A^3+B^3}{A^2-AB+B^2}$.)

   (c) (10 pts) Sketch the graph of $y = f(x)$. Indicate clearly in your sketch, if any, where it is increasing/decreasing, where it concaves upward/downward, all relative maxima/minima, inflection points and asymptotes.

   (d) (6 pts) A particle is moving along the curve $y = f(x)$. If the rate of change of its x-coordinate is 1 unit/s. Find the rate of change of its distance from the origin at the point $(2, 2^{\frac{4}{3}})$.

試題必須隨卷繳回